Coordination of Distributed Energy Resources in Lossy Networks for Providing Frequency Regulation

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***Abstract*—In this paper, we address the problem of optimally coordinating the response of a set of distributed energy resources (DERs) in a lossy distribution feeder in order to provide frequency regulation services to the bulk grid to which the feeder is connected. The problem objective is to minimize the incremental total network losses, which are approximated by a linear function of the incremental changes in active power provided by the DERs. The amount by which each DER can change its active power is constrained to lie in some interval. In addition, the DER incremental power changes must be such that the incremental change in power at the feeder head is equal to some prescribed value. In order for the DERs to determine their optimal contribution, we propose an iterative distributed algorithm that relies on local computations performed by each DER based on information acquired from other geographically-close DERs. We illustrate the algorithm performance via numerical experiments in a distributed computing platform.**

I. Introduction

In a power system, frequency regulation services are utilized to track the moment-to-moment load fluctuations in near real-time, and ensure generation and load are balanced in each control area [1]. Traditionally, frequency regulation services are mainly provided by large synchronous generators. On the other hand, it has been acknowledged that distributed energy resources (DERs), such as energy storage, photovoltaic systems, and plug-in vehicles, could potentially be utilized to provide such services [2], [3]. In this regard, since the frequency regulation capability of individual DERs is typically small, it is necessary to coordinate the response of many of them, possibly located at different buses of the distribution feeder. The coordination must be such that the incremental change (with respect to some nominal value) in the power exchanged by the distribution feeder with the rest of the system matches a request made by the Independent System Operator (ISO), which we refer to as the regulation power. Then, the problem is to optimally allocate the regulation power among the set of DERs subject to some upper and lower limits on the incremental change of each DER; we refer to this problem as the optimal DER coordination problem (ODCP).

In a market environment, the collection of DERs is incentivized to provide the exact amount of the requested regulation power at the feeder head since otherwise they will incur penalty or loss of performance payments [4], [5]. As such, it is necessary to explicitly take into account the impacts of the network losses when determining the regulation power provided by each DER. In fact, it is desired to optimally dispatch the DERs such that the exact requested regulation power at the feeder head is provided with minimum incremental total network losses. Yet, the total network losses are a nonlinear function of the voltage magnitudes and angles, as well as the network parameters; this complicates the ODCP formulation.

A practical approach to model network losses is to use loss factors (LFs) — linear sensitivities of the total network losses with respect to changes in power injections. LFs can be computed using the power flow model [6], [7], or estimated using measurements obtained from phasor measurement units (PMUs) [8], [9]. In this paper, we use LFs to formulate the ODCP as a linear program, the objective of which is to minimize the total incremental network losses, which can approximated as a linear function of the DER incremental active power changes. In addition, DER incremental power changes are constrained to lie in some interval, and must be such that the incremental change in power at the head of the feeder to which the DERs are connected is equal to some prescribed value.

The aforementioned linear program can be solved by a centralized decision maker that has complete and up-to-date information of each DER, as well as the network parameters. However, this requires bidirectional information exchange between the centralized decision maker and every individual DER. All these requirements, however, result in a heavy burden on the communication network, and in the presence of a single communication link failure, the centralized decision maker may fail to find the optimal solution. As such, it is more desirable to develop a distributed decision-making scheme such that each DER can determine its optimal output with minimal information exchange with a subset of DERs.

In this paper, we pursue the distributed approach discussed above, and by leveraging the structural characteristics of the ODCP, we devise a scheme for computing the solution to the problem that is amenable for a distributed implementation. For realizing this distributed implementation, we leverage the so-called ratio-consensus algorithm [10], and its variants that take into account several issues that are important for a practical implementation [11].

The remainder of this paper is organized as follows. Section II presents the ODCP formulation, followed by a description of the DER communication model, and a brief description of the basic ratio-consensus algorithm. Section III develops a centralized solution to the ODCP, and a scheme for distributing the computation of such solution. The performance of the proposed distributed algorithm is demonstrated in Section IV and the concluding remarks are presented in Section V.

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II. Preliminaries

In this section, we first present the formulation of the ODCP in the context of frequency regulation in a lossy network. Then, the communication model of the DERs is described. The ratio consensus algorithm, based on which the distributed solution to the ODCP is proposed, is also briefly reviewed.

A. Optimal DER Coordination Problem

Let $\mathcal{X} = \{1, \cdots, N\}$ be the set of DERs connected to some distribution network (feeder). Let x_j denote the regulation power provided by DER j, $j \in \mathcal{X}$, where $x_j \geq 0$ if it results in a non-negative change in the power injected into the distribution network, and $x_j < 0$ otherwise. Let $\underline{x}_j$ and $\overline{x}_j$ denote the minimum and maximum values of x_j, $j \in \mathcal{X}$, respectively. Without loss of generality, assume $\underline{x}_j < 0 < \overline{x}_j$.

Assume that the distribution network is interconnected via some tie-lines to a bulk power system. Let x_0 denote the incremental change in active power, with respect to some nominal value, flowing across the aforementioned tie-lines, where $x_0 \geq 0$ if it results in a non-negative change in the power injected into the distribution network, and $x_0 < 0$ otherwise.

Let X denote some regulation signal sent by the ISO, then the objective is to determine the x_j's so that $x_0 = -X$. In a lossy distribution network, the total regulation power provided by the DERs, $\sum_{j=1}^{N} x_j$, must exceed X to compensate for losses. The incremental total network losses associated with the x_j's, denoted by $\Delta\ell(x_1, \cdots, x_N)$, can be approximated by

$$\Delta\ell(x_1, \cdots, x_N) = \sum_{j=1}^{N} \Lambda_i x_j, \tag{1}$$

where $\Lambda_j < 1$ is the LF associated with DER j defined to be the partial derivative of the total network losses with respect to the active power injection from DER i evaluated at the operating point corresponding to the DERs' nominal active power outputs. Let $\mu_j = \frac{\Lambda_j}{1-\Lambda_j}$, $j \in \mathcal{X}$, and define a multiset $\mathcal{M} = \{\mu_1 \cdots, \mu_N\}$. Our objective here is to minimize the incremental total network losses. Then, given $\underline{x}_j$, $\overline{x}_j$, Λ_j, and X, the optimal choice for the x_j's can be found by solving the following problem:[1]

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{j=1}^{N} \Lambda_j x_j \\ \text{subject to} \quad & \textstyle\sum_{j=1}^{N} (1-\Lambda_j) x_j = X, \\ & \underline{x}_j \leq x_j \leq \overline{x}_j, \forall j \in \mathcal{X}. \end{aligned} \tag{2}$$

B. DER Communication Model

The communication network that enables the information exchange between DERs can be described by a strongly connected directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \cdots, N\}$ is the vertex set (each vertex/node corresponds to a DER), and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, where $(j, i) \in \mathcal{E}$ if node j can receive information from node i. We allow each node to receive information from itself, i.e., $(j, j) \in \mathcal{E}$ for any $j \in \mathcal{V}$. All nodes that can transmit information to and receive information from node j are referred to as the in-neighbors and the out-neighbors of node j, respectively. The set of in-neighbors of node j (including itself) is denoted by $\mathcal{N}_j^-$, whereas the set of out-neighbors is denoted by $\mathcal{N}_j^+$. The in-degree and the out-degree of node j are, respectively, $D_j^- = |\mathcal{N}_j^-|$ and $D_j^+ = |\mathcal{N}_j^+|$, where $|\cdot|$ denotes the cardinality of a set. In addition, denote by $d(\mathcal{G})$ the diameter of a strongly connected graph $\mathcal{G}$, i.e., the length of the longest among all shortest paths connecting any pair of nodes.

C. Ratio Consensus Algorithm

Assume that the communication network enabling the information exchange between DERs conforms to a strongly connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ as described in Section II-B. In ratio consensus, each node j maintains two internal states, denoted by y_j and z_j, and updates them iteratively. Let $y_j[k]$ and $z_j[k]$ denote the respective values of y_j and z_j at iteration k, $k = 0, 1, \cdots$, which are updated as follows:

$$\begin{aligned} y_j[k+1] &= \sum_{l \in \mathcal{N}_j^-} \tfrac{1}{D_l^+} y_l[k], \\ z_j[k+1] &= \sum_{l \in \mathcal{N}_j^-} \tfrac{1}{D_l^+} z_l[k]. \end{aligned} \tag{3}$$

Assuming $z_j[k] > 0$, $\forall j \in \mathcal{X}$, node j computes

$$\gamma_j[k] = \frac{y_j[k]}{z_j[k]}. \tag{4}$$

Then as shown in [10], [12], it follows:

$$\gamma^\star := \lim_{k \to \infty} \gamma_j[k] = \frac{\sum_{l=1}^{N} y_l[0]}{\sum_{l=1}^{N} z_l[0]}, \ \forall j \in \mathcal{X}. \tag{5}$$

We will show later that ratio consensus is a primitive to solve the ODCP in a distributed fashion. In this regard, it is necessary to stop the execution of the ratio consensus algorithm after a finite number of iterations, k_0. Next, we present a distributed algorithm that accomplishes this, while guaranteeing that $|\gamma_j[k_0] - \gamma^\star|$ is smaller than some $\epsilon > 0$, $\forall j \in \mathcal{X}$.

Assume node j maintains two additional internal states denoted by M_j and m_j. Let $M_j[k]$ and $m_j[k]$ denote the respective values of M_j and m_j at iteration k, which are updated using the max- and min-consensus as follows:

$$\begin{aligned} M_j[k+1] &= \max_{l \in \mathcal{N}_j^-} M_l[k], \\ m_j[k+1] &= \min_{l \in \mathcal{N}_j^-} m_l[k]. \end{aligned} \tag{6}$$

$M_j[k]$ and $m_j[k]$ are periodically reinitialized every $d(\mathcal{G})$ iterations. Specifically, $M_j[k]$ and $m_j[k]$ are reinitialized to be $\gamma_j[k]$ when $k \mod d(\mathcal{G}) = 0$.

The authors in [11] showed that for any $k \mod d(\mathcal{G}) = 0$, $M_j[k + d(\mathcal{G})]$ will reach $\max\limits_{l \in \mathcal{V}} M_l[k]$, and $m_j[k + d(\mathcal{G})]$ will

[1]If each DER has a cost associated with its power generation, the objective function in (2) can be chosen to reflect the incremental total cost associated with x_j, $j \in \mathcal{X}$. Typically, the generation cost is quadratic, in which case the incremental cost is a linear function of the x_j's. Yet, changing the objective function from one linear function to another does not change the nature of the problem and thus can be solved in a similar manner.

reach $\min_{l\in\mathcal{V}} m_l[k]$, which are essentially the respective maximum and minimum values of $\gamma_j[k]$ for $j \in \mathcal{X}$. Before reinitializing M_j and m_j, each node checks if the worst-case error between $\gamma_j[k]$ and $\gamma^\star$, given by $M_j[k+d(\mathcal{G})]-m_j[k+d(\mathcal{G})]$, is smaller than ϵ. The ratio consensus algorithm will stop at iteration k_0 when

$$M_j[k_0] - m_j[k_0] < \epsilon. \tag{7}$$

III. Solution to the Optimal DER Coordination Problem

In this section, we first formulate the dual problem of the ODCP and propose a method to solve it. Then, we show how such a method can be implemented in a distributed fashion using ratio consensus.

A. Centralized Solution

The Lagrange dual problem of (2) is

$$\max_{\mu} \inf_{x_j\in[\underline{x}_j,\overline{x}_j]} \sum_{j=1}^{N} \Lambda_j x_j - \mu(\sum_{j=1}^{N}(1-\Lambda_j)x_j - X), \tag{8}$$

which can be rewritten as

$$\max_{\mu} \mu X - \sum_{j=1}^{N} f_j(\mu), \tag{9}$$

where $f_j(\mu), \forall j$, is a piecewise linear function of the following form:

$$f_j(\mu) = \begin{cases} (1-\Lambda_j)\underline{x}_j\mu - \Lambda_j\underline{x}_j, & \mu \le \mu_j, \\ (1-\Lambda_j)\overline{x}_j\mu - \Lambda_j\overline{x}_j, & \mu > \mu_j. \end{cases} \tag{10}$$

Assuming the primal problem is feasible, by strong duality, there exists μ^* that solves (9). We next show how to determine μ^*.

Let $h(\mu) := \sum_{j=1}^{N} h_j(\mu)$, where $h_j(\mu)$ is a two-valued step function defined as follows:[2]

$$h_j(\mu) = \begin{cases} (1-\Lambda_j)\underline{x}_j, & \text{if } \mu \le \mu_j, \\ (1-\Lambda_j)\overline{x}_j, & \text{if } \mu > \mu_j. \end{cases} \tag{11}$$

Under the assumption that (2) is feasible, we can compute μ^* as follows:

$$\mu^* = \underset{\mu\in\mathcal{M},\, \frac{h(\mu)}{X}\le 1}{\arg\min} \left| \frac{h(\mu)}{X} - 1 \right|. \tag{12}$$

The detailed proof of this result is given in the Appendix.

Given μ^*, we can then determine the optimal solution to the primal problem. First, partition $\mathcal{X}$ as follows:

$$\begin{aligned} \mathcal{X}^+ &= \{j \in \mathcal{X} : \mu_j > \mu^*\}, \\ \mathcal{X}^0 &= \{j \in \mathcal{X} : \mu_j = \mu^*\}, \\ \mathcal{X}^- &= \{j \in \mathcal{X} : \mu_j < \mu^*\}. \end{aligned} \tag{13}$$

[2]It is easy to check that $h_j(\mu)$ is one of subgradients of $f_j(\mu)$ and each of its function values is the derivative of some line segment of $f_j(\mu)$.

The DERs in $\mathcal{X}^+$ ($\mathcal{X}^-$) are the more (less) "costly" units (also referred to as the non-marginal DERs), and their outputs are set as follows:

$$x_j^* = \begin{cases} \underline{x}_j, & \forall j \in \mathcal{X}^+, \\ \overline{x}_j, & \forall j \in \mathcal{X}^-. \end{cases} \tag{14}$$

The DERs in $\mathcal{X}^0$ are referred to as the marginal DERs. If $\mathcal{X}^0$ is a singleton, if follows from the equality constraint in (2) that the output of DER $j \in \mathcal{X}^0$ must be set to:

$$x_j = \frac{X - \sum_{l\in\mathcal{X}^+}(1-\Lambda_l)\underline{x}_l - \sum_{l\in\mathcal{X}^-}(1-\Lambda_l)\overline{x}_l}{1-\Lambda_j}. \tag{15}$$

If $\mathcal{X}^0$ is not a singleton, the solution to the ODCP may not be unique. In order to determine a solution, we need a policy to allocate among the marginal DERs the difference between the requested regulation power, X, and the total regulation power provided by non-marginal DERs. In this paper, we adopt a "fair splitting" policy, in which the aforementioned difference is allocated as follows:

$$x_j^* = \underline{x}_j + \alpha(\overline{x}_j - \underline{x}_j), \tag{16}$$

where

$$\alpha = \frac{X - \sum_{l\in\mathcal{X}^+\cup\mathcal{X}^0}(1-\Lambda_l)\underline{x}_l - \sum_{l\in\mathcal{X}^-}(1-\Lambda_l)\overline{x}_l}{\sum_{l\in\mathcal{X}^0}(1-\Lambda_l)(\overline{x}_l - \underline{x}_l)}. \tag{17}$$

We note that (16) and (17) are reduced to (15) when $\mathcal{X}^0$ is a singleton.

To sum up, solving the ODCP involves two steps. First, find μ^* based on (12); then compute x_j, $\forall j \in \mathcal{X}$, using (14) and (16). In principle, both steps require a centralized decision maker that has access to information from all DERs. However, we will show next that the solution method described above can be implemented in a distributed fashion.

B. Distributed Solution

Assume Λ_1 to Λ_N, and correspondingly, $\mathcal{M}$, is known to all nodes. Each node will be able to determine μ^* based on (12) after it learns the value of $\frac{h(\mu)}{X}$, $\forall \mu \in \mathcal{M}$. Given μ^*, the optimal regulation power of the non-marginal DERs can be readily determined based on (13) and (14). The learning of $\frac{h(\mu)}{X}$ can be done distributedly through ratio consensus. Each node j executes N copies of the ratio consensus numerator iteration with $y_j^{(i)}[0] = h_j(\mu_i), \forall i \in \mathcal{X}$. Meanwhile, each node j also executes one copy of the ratio consensus denominator iteration with $z_1[0] = X$ and $z_j[0] = 0, \forall j \in \mathcal{X} \setminus \{1\}$. Updating $y_j^{(i)}[k]$ and $z_j[k]$ according to (3), by (5), each node j will asymptotically learn $\frac{h(\mu_i)}{X}$, $\forall i \in \mathcal{X}$, as follows:

$$\frac{h(\mu_i)}{X} = \frac{\sum_{l\in\mathcal{X}} h_l(\mu_i)}{X} = \lim_{k\to\infty} \frac{y_j^{(i)}[k]}{z_j[k]}. \tag{18}$$

In practice, the ratio consensus needs to be stopped after a finite number of iterations; this can be accomplished using the max- and min-consensus. Each node executes N copies of the max-consensus iteration with $M_j^{(i)}[0] = 2\epsilon$, and N copies of the min-consensus iteration with $m_j^{(i)}[0] = 0$, $\forall i \in \mathcal{X}$. Each node checks if $M_j[k] - m_j[k] < \epsilon$ when $k \mod d(\mathcal{G}) = 0$; if not, $M_j^{(i)}[k]$ and $m_j[k]$ are reinitialized to be $\gamma_j[k]$. Updating $M_j^{(i)}[k]$ and $m_j[k]$ based on (6), when the consensus iteration stops, the error between the computed value of μ^* by each node and its true value is guaranteed to be bounded by ϵ. The detailed procedures of using ratio consensus to distributedly compute μ^* is presented in Algorithm 1.

Algorithm 1 Distributed Computation of μ^*

Input: X (known to node 1) and error bound ϵ (known to all nodes)

Output: μ^*

initialize: each node j maintains $3N+1$ variables, $y_j^{(1)}, \cdots, y_j^{(N)}$, z_j, $M_j^{(1)}, \cdots, M_j^{(N)}$, $m_j^{(1)}, \cdots, m_j^{(N)}$, and initializes them as follows:

$$y_j^{(i)}[0] = h_j(\mu_i), \forall i, j \in \mathcal{X},$$
$$z_1[0] = X,$$
$$z_j[0] = 0, \forall j \in \mathcal{X} \setminus \{1\},$$
$$M_j^{(i)}[0] = 2\epsilon, m_j^{(i)}[0] = 0, \forall i, j \in \mathcal{X}$$

set iteration index: $k = 0$
while true do
  if $k \mod d(\mathcal{G}) = 0$ **then**
    if $M_j^{(i)}[k] - m_j^{(i)}[k] < \epsilon$ **then**
      break
    else
      if $k \geq d(\mathcal{G})$ **then**
        $M_j^{(i)}[k] = m_j^{(i)}[k] = \gamma_j^{(i)}[k]$
      end if
    end if
  end if
  broadcast state variables to out-neighbors
  receive state variables from in-neighbors
  update state variables:

$$y_j^{(i)}[k+1] = \sum_{l \in \mathcal{N}_j^-} \frac{1}{D_l^+} y_l^{(i)}[k],$$
$$z_j[k+1] = \sum_{l \in \mathcal{N}_j^-} \frac{1}{D_i^+} z_l[k],$$
$$M_j^{(i)}[k+1] = \max_{l \in \mathcal{N}_j^-} M_l^{(i)}[k],$$
$$m_j^{(i)}[k+1] = \min_{l \in \mathcal{N}_j^-} m_l^{(i)}[k]$$

  compute $\gamma_j^{(i)}[k+1] = \dfrac{y_j^{(i)}[k+1]}{z_j[k+1]}$
end while
compute $\mu^* = \arg\min_{\mu \in \mathcal{M}, \frac{h(\mu)}{X} \leq 1} \left| \dfrac{h(\mu)}{X} - 1 \right|$, where $\dfrac{h(\mu_i)}{X} = \gamma_j^{(i)}[k+1]$

To determine the optimal regulation power of the marginal DERs, the corresponding nodes need to learn the value of α defined in (17). Given α, the optimal regulation power of the marginal DERs can be computed according to (16). The value that α takes is the ratio of two sums, the terms in which are known locally to the nodes. As such, similar to the computation of μ^*, α can be obtained in a distributed fashion using ratio consensus. The procedures of distributedly computing α are detailed in Algorithm 2.

Algorithm 2 Distributed Computation of α

Input: : X (known to node 1) and error bound ϵ (known to all nodes)

Output: : α

initialize: each node j maintains 4 variables, y_j, z_j, M_j, m_j, and initializes them as follows:

$$y_1[0] = X - (1 - \Lambda_1)\underline{x}_1, \text{if } 1 \in \mathcal{X}^+ \cup \mathcal{X}^0$$
$$y_1[0] = X - (1 - \Lambda_1)\overline{x}_1, \text{if } 1 \in \mathcal{X}^-$$
$$y_j[0] = -(1 - \Lambda_j)\underline{x}_j, \forall j \in \mathcal{X}^+ \cup \mathcal{X}^0,$$
$$y_j[0] = -(1 - \Lambda_j)\overline{x}_j, \forall j \in \mathcal{X}^-,$$
$$z_j[0] = (1 - \Lambda_j)(\overline{x}_j - \underline{x}_j), \forall j \in \mathcal{X}^0,$$
$$z_j[0] = 0, \forall j \notin \mathcal{X}^0,$$
$$M_j[0] = 2\epsilon, m_j[0] = 0, \forall j \in \mathcal{X}$$

set iteration index: $k = 0$
while true do
  if $k \mod d(\mathcal{G}) = 0$ **then**
    if $M_j[k] - m_j[k] < \epsilon$ **then**
      break
    else
      if $k \geq d(\mathcal{G})$ **then**
        $M_j[k] = m_j[k] = \gamma_j[k]$
      end if
    end if
  end if
  broadcast state variables to out-neighbors
  receive state variables from in-neighbors
  update state variables:

$$y_j[k+1] = \sum_{l \in \mathcal{N}_j^-} \frac{1}{D_l^+} y_l[k],$$
$$z_j[k+1] = \sum_{l \in \mathcal{N}_j^-} \frac{1}{D_i^+} z_l[k],$$
$$M_j[k+1] = \max_{l \in \mathcal{N}_j^-} M_l[k],$$
$$m_j[k+1] = \min_{l \in \mathcal{N}_j^-} m_l[k]$$

  compute $\gamma_j[k+1] = \dfrac{y_j[k+1]}{z_j[k+1]}$
end while
compute $\alpha = \gamma_j^{(i)}[k+1]$

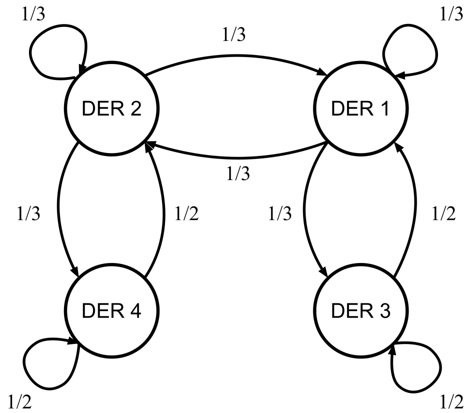

Fig. 1. DER communication graph in the experiment.

IV. Experimental Results

To demonstrate the effectiveness of the proposed distributed algorithm, we utilize a distributed computing system developed in our earlier work on microgrid control [13]. In this system, there are four nodes, each comprised of an Arduino Mega micro-controller and an XBee module. The Arduino Mega micro-controller allows each node to perform computations based on information acquired from other nodes via the Xbee module. In our experiments, we associate each node in this system with a unique DER, with the graph in Fig. 1 describing the information exchange among them.

The DER maximum regulation power and minimum regulation power are as follows: $\overline{x}_1 = 0.3$, $\overline{x}_2 = 0.8$, $\overline{x}_3 = 0.5$, $\overline{x}_4 = 0.4$, and $\underline{x}_1 = -0.3$, $\underline{x}_2 = -0.8$, $\underline{x}_3 = -0.5$, $\underline{x}_4 = -0.4$. The LFs are as follows: $\Lambda_1 = 0$, $\Lambda_2 = 0.01$, $\Lambda_3 = 0.02$, $\Lambda_4 = 0.04$; as a result, $\mathcal{M} = \{0, \frac{1}{99}, \frac{1}{49}, \frac{1}{24}\}$. Assume each node knows $\mathcal{M}$ before the execution of Algorithm 1.

Node 1 receives regulation signal from the bulk grid operator. Each node can communicate with itself. In addition, there is bidirectional communication between nodes 1 and 2, 1 and 3, and 2 and 4, as illustrated in Fig. 1. The out-degrees of nodes 1 and 2 are 3 while those of nodes 3 and 4 are 2. Let $X = 1.8$ and $\epsilon = 0.0001$.

The consensus results when node 1 computes μ^* are shown in Figs. 2 to 4. The ratios, $\gamma_j^{(i)}$, are approximate values for $\frac{h(\mu_i)}{X}$. The evolution of the max- and min-consensus results are also presented. The estimated values of $\frac{h(\mu_1)}{X}$, $\frac{h(\mu_2)}{X}$, $\frac{h(\mu_3)}{X}$, $\frac{h(\mu_4)}{X}$ by all nodes converge to $-1.0922, -0.7590, 0.1210, 0.6660$, respectively, after 35 iterations.

Based on this result, all nodes will agree on the value of μ^* to be $\frac{1}{24}$. This result is indeed intuitively correct since DER 4 has the largest LF, yet the maximum total regulation power provided by DERs 1 to 3 cannot meet the regulation requirement.

After each node learns μ^*, the regulation power of DERs 1 to 3 will be set to their corresponding maximum. Yet, the regulation power of DER 4 still needs to be determined through another round of consensus using Algorithm 2, which essentially computes α as introduced earlier. The consensus results when computing α are shown in Figs. 5 to 7.

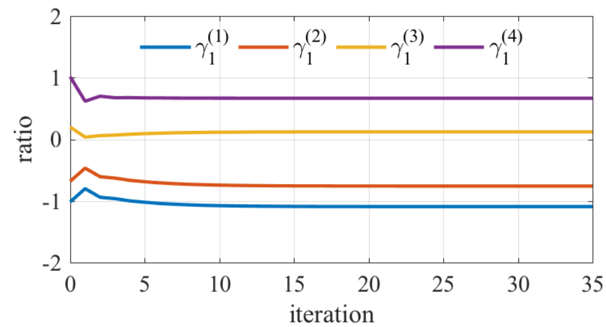

Fig. 2. Evolution of ratio consensus when node 1 computes μ^*.

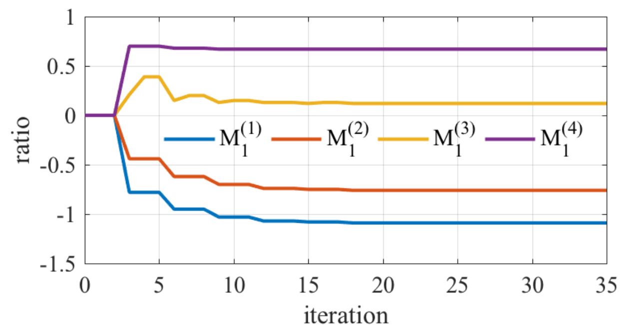

Fig. 3. Evolution of max-consensus when node 1 computes μ^*.

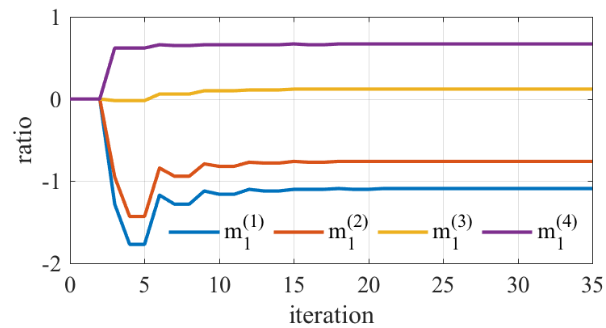

Fig. 4. Evolution of min-consensus when node 1 computes μ^*.

All nodes reach an agreement on the value of α to be 0.7840 after 38 iterations. As such, the regulation power of DER 4 will be set to 0.2272, which is computed based on (16). It can be easily verify that the total provided regulation power in the presence of losses is 1.8 when DERs set their regulation power to 0.3, 0.8, 0.5, and 0.2272, respectively.

V. Concluding Remarks

In this paper, we first developed a solution to the ODCP in the context of frequency regulation in a lossy distribution network, where total network losses are approximated using LFs. We further proposed a ratio consensus-based algorithm for the distributed computations of the solution. The proposed

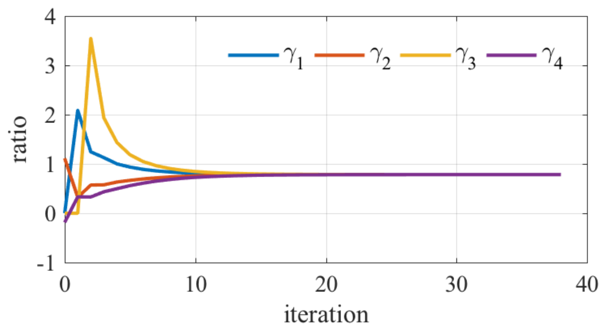

Fig. 5. Evolution of ratio consensus when computing α.

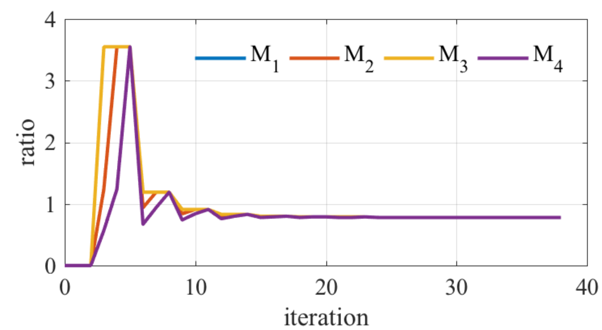

Fig. 6. Evolution of max-consensus when computing α.

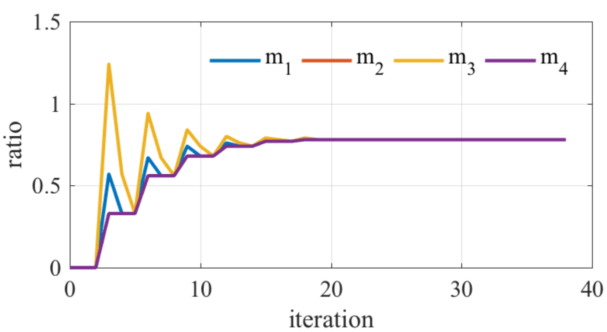

Fig. 7. Evolution of min-consensus when computing α.

distributed algorithm is computationally efficient and easy to implement as evidenced by our experimental results.

A key parameter in the formulation of the ODCP is the LFs, which we assume are known to the nodes in this work. Yet, the LFs will change when power injections or the topology of the distribution network change. As such, the LFs need to be computed before they are used in the ODCP. Future works will focus on the development of online estimation methods that will provide accurate LFs.

Appendix
Proof of Solution to The Dual Problem

Assume the ODCP is feasible, then by strong duality, there exits μ^* that solves (9). Without loss of generality, assume $\Lambda_1 \leq \cdots \leq \Lambda_N < 1$. Then, $\mu_1 \leq \cdots \leq \mu_N$. Define $\mu_0 = -\infty$ and $\mu_{N+1} = \infty$. There exists some $k \in \{0, 1, \cdots, N\}$ such that $\mu_k < \mu^* \leq \mu_{k+1}$. As a result, $\mathcal{X}^- = \{1, \cdots, k\}$ and $\mathcal{X}^0 \cup \mathcal{X}^+ = \{k+1, \cdots, N\}$. It can be easily verified that $h(\mu_{k+1}) = h(\mu^*)$. The objective function value of the dual problem (9) is

$$\mu^* X - \sum_{j=1}^{N} f_j(\mu^*) = \mu^* X$$
$$-\sum_{j=1}^{k}((1-\Lambda_j)\overline{x}_j\mu^* - \Lambda_j\overline{x}_j) - \sum_{j=k+1}^{N}((1-\Lambda_j)\underline{x}_j\mu^* - \Lambda_j\underline{x}_j)$$
$$= \mu^*(X - h(\mu^*)) + \sum_{j=1}^{k}\Lambda_j\overline{x}_j + \sum_{j=k+1}^{N}\Lambda_j\underline{x}_j.$$

The solution to the primal problem is

$$x_j^* = \begin{cases} \underline{x}_j, & \forall j \in \mathcal{X}^+, \\ x_j^0, & \forall j \in \mathcal{X}^0, \\ \overline{x}_j, & \forall j \in \mathcal{X}^-, \end{cases}$$

which must satisfy the following equality constraint:

$$\sum_{j\in\mathcal{X}^-}(1-\Lambda_j)\overline{x}_j + \sum_{j\in\mathcal{X}^0}(1-\Lambda_j)x_j^0 + \sum_{j\in\mathcal{X}^+}(1-\Lambda_j)\underline{x}_j = X.$$

Since $\underline{x}_j \leq x_j^0 \leq \overline{x}_j$, and

$$h(\mu_{k+1}) = \sum_{j=1}^{k}(1-\Lambda_j)\overline{x}_j + \sum_{j=k+1}^{N}(1-\Lambda_j)\underline{x}_j \leq X,$$
$$h(\mu_{k+2}) = \sum_{j=1}^{k+1}(1-\Lambda_j)\overline{x}_j + \sum_{j=k+2}^{N}(1-\Lambda_j)\underline{x}_j \geq X.$$

Therefore,

$$\underset{\mu\in\mathcal{M}, h(\mu)\leq X}{\arg\min} |h(\mu) - X| = \mu_{k+1}.$$

Since

$$\mu_{k+1}X - \sum_{j=1}^{N} f_j(\mu_{k+1}) = \mu_{k+1}X$$
$$-\sum_{j=1}^{k}((1-\Lambda_j)\overline{x}_j\mu_{k+1} - \Lambda_j\overline{x}_j) - \sum_{j=k+1}^{N}((1-\Lambda_j)\underline{x}_j\mu_{k+1} - \Lambda_j\underline{x}_j)$$
$$= \mu_{k+1}(X - h(\mu_{k+1})) + \sum_{j=1}^{k}\Lambda_j\overline{x}_j + \sum_{j=k+1}^{N}\Lambda_j\underline{x}_j,$$

then the following relation holds:

$$\mu_{k+1}X - \sum_{j=1}^{N} f_j(\mu_{k+1}) \geq \mu^* X - \sum_{j=1}^{N} f_j(\mu^*).$$

However, by the definition of μ^*,

$$\mu^* X - \sum_{j=1}^{N} f_j(\mu^*) \geq \mu_{k+1}X - \sum_{j=1}^{N} f_j(\mu_{k+1}).$$

Thus, $\mu^* = \mu_{k+1}$, and (12) gives the solution to the dual problem (9).

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