Convex Relaxation of OPF in Multiphase Radial Networks with Delta Connection

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***Abstract*—This paper focuses on multiphase radial distribution networks with wye and delta connections, and proposes a semidefinite relaxation of the AC optimal power flow (OPF) problem. Two multiphase power flow models are developed to facilitate the integration of delta-connected loads or generation resources in the OPF problem. The first model is referred to as the extended branch flow model (EBFM). The second model leverages a linear relationship between phase-to-ground power injections and delta connections that holds under a balanced voltage approximation (BVA). Based on these models, pertinent OPF problems are formulated and relaxed to semidefinite programs (SDPs). Numerical studies on IEEE test feeders show that the proposed SDP relaxations can be solved efficiently by a generic optimization solver. Numerical evidence also indicates that solving the resultant SDP under BVA is faster than under EBFM. Moreover, both SDP solutions are numerically exact with respect to voltages and branch flows. It is further shown that the SDP solution under BVA has a small optimality gap, and the BVA model is accurate in the sense that it reproduces actual system voltages.**

I. Introduction

AC optimal power flow (OPF) is a fundamental problem in power system operations. At the distribution level, OPF underlies (and possibly unifies) many applications, such as Volt/VAr/Watt control, dispatch of renewable energy sources, and demand response. With the rapid growth of distributed energy resources—including renewables, energy storage devices, and flexible loads—it is crucial for distribution systems to solve OPF in a fast and scalable way over a large number of active nodes. Towards this end, nonconvexity of AC OPF is a major hurdle, and recent efforts have looked at centralized and distributed OPF solution methods based on convex approximations or relaxations; see, for example, [1]–[7] and pertinent references therein.

A popular convex approximation is obtained through the linearization of the power flow equations, using DC power flow [8], LinDistFlow [9], [10], or recently developed methods such as [11]–[14]. In particular, power flow linearization methods in multiphase networks are proposed in [10], [12], [14], where [10], [12] are applicable to networks with wye connections, and [14] incorporates delta connections as well.

Semidefinite relaxation is another commonly taken approach to convexify OPF problems. To the best of our knowledge, it was first proposed in [15] to solve OPF as a semidefinite program (SDP) in single-phase networks with general topologies, and it was first studied in [16] whether and when this SDP relaxation is exact. Sparsity of power networks was exploited to simplify the SDP relaxation [17], [18], and relaxation to a more efficiently solvable second-order cone program (SOCP) is available in radial (tree) networks [19]. Techniques such as quadratic convex relaxation [20], moment/sum-of-squares hierarchy [21], and strong SOCP relaxation [22] have been explored to strengthen the SDP relaxation. See [23], [24] for a survey of convex relaxations of AC OPF in single-phase distribution networks. In multiphase radial networks, [3] was the first that we know of that applied SDP relaxation; later, [10] illustrated SDP relaxation on a numerically more stable branch flow model. Based on these studies, distributed OPF algorithms were developed in [3], [6].

Distribution networks in practice are not only multiphase and radial but also composed of both wye and delta connections [25], [26]. Oftentimes, wye- and delta-connected loads or resources can be present at the same time at the secondary of a distribution transformer. A substantial body of literature discusses power flow models, formulations, algorithms, and analyses with wye and delta connections [27]–[29]; however, not much work on OPF has explicitly incorporated delta connections. In [30], OPF under wye and delta connections was formulated and solved as a mixed-integer nonlinear program without dealing with the nonconvexity issue. The SDP relaxation-based OPF studies— [3], [6], [10]—all take phase-to-ground power injections as optimization variables, and hence essentially assume only wye connections exist in the network.

In this paper, we propose an SDP relaxation of the AC OPF problem in multiphase radial networks with wye and delta connections. To facilitate the derivation of an SDP relaxation, we develop two power flow models to incorporate delta connections. The first, hereafter referred to as extended branch flow model (EBFM), extends the branch power flow model of [10] to account for delta connections. The second, hereafter referred to as balanced voltage approximation (BVA), exploits a linear relationship between phase-to-ground power injections and delta connections that holds approximately when three-phase voltages are nearly balanced. Formulations and SDP relaxations of OPF under both models are presented. Numerical studies on IEEE 13- and 37-node networks [26] show that both SDP relaxations can be solved efficiently by a generic optimization solver (e.g., SeDuMi [31]). It is also noticed that solving SDP under BVA is faster than under EBFM. Moreover, both solutions are numerically exact with respect to voltages and branch flows. The solution under EBFM is not exact with respect to the delta-connected variables, but it provides a lower bound of the OPF objective and reveals a small optimality gap of the solution under BVA. Finally, accuracy of the BVA model is shown via comparisons between voltages recovered from SDP and solved by OpenDSS [32].

The rest of the paper is organized as follows. Section II introduces the model of multiphase radial networks with both wye and delta connections. Sections III and IV present two power flow models, EBFM and BVA, respectively, and formulate the OPF problems and their SDP relaxations under these two models. Section V shows numerical results. Section VI concludes the paper and discusses future work.

II. Multiphase Radial Network Model

A. Notation

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ denote the set of real, complex, and (nonzero) natural numbers, respectively. Define $\mathrm{j} := \sqrt{-1}$. For $n \in \mathbb{N}$, let $\mathbb{H}^{n\times n}$ denote the space of all n-by-n Hermitian matrices. For any scalar, vector, or matrix A, let A^T, A^*, and A^H denote its transpose, element-wise conjugate, and conjugate transpose, respectively. For a square matrix A, let $\mathrm{diag}(A)$ denote the column vector composed of its diagonal entries, and $\mathrm{tr}(A)$ denote its trace. For an index set S, let A_S denote the set $\{A_i \mid i \in S\}$, or (when A_i's are scalars) the column vector composed of $\{A_i \mid i \in S\}$. Further, given an n-by-n matrix A, $\mathrm{rank}(A)$ returns the rank of A. Let $\mathbf{1}_S$ ($\mathbf{0}_S$) denote the $|S|$-dimensional column vector with all elements 1 (0), where $|S|$ denotes the cardinality of S. The subscript S is omitted when its meaning is clear from the context.

B. Network Model

Let $\mathcal{N} = \{0, 1, \ldots, n\}$ denote the set of buses (nodes) of a radial multiphase distribution network. Let 0 index the substation (or point of common coupling), and define $\mathcal{N}^+ := \mathcal{N} \setminus \{0\}$. Let $\mathcal{E}$ denote the set of lines connecting the buses. In particular, each line connects an ordered pair (i, j) of buses, where bus i lies between bus 0 and bus j. We use $(i, j) \in \mathcal{E}$ and $i \to j$ interchangeably, and denote $i \sim j$ if either $i \to j$ or $j \to i$. Let $\mathcal{N}_{\text{leaf}}$ denote the set of "leaf" buses from which there are no directed lines.

For ease of exposition and notational simplicity, we assume that all the buses $i \in \mathcal{N}$ and lines $(i, j) \in \mathcal{E}$ have three phases: a, b, c; and define $\Phi := \{a, b, c\}$ and $\Phi' := \{ab, bc, ca\}$. In Section III-B, we will discuss how missing phases on certain buses and lines can be readily managed. For $i \in \mathcal{N}$ and $\phi \in \Phi$, let V_i^ϕ denote the complex voltage on phase ϕ of bus i, and define $V_i := [V_i^a, V_i^b, V_i^c]^T$. For $i \sim j$ and $\phi \in \Phi$, let I_{ij}^ϕ denote the phase ϕ current on the line from bus i to bus j, and define $I_{ij} := [I_{ij}^a, I_{ij}^b, I_{ij}^c]^T$. Let $y_i \in \mathbb{C}^{3\times 3}$ denote the shunt admittance at bus i, and $z_{ij} \in \mathbb{C}^{3\times 3}$ denote the series impedance of line $i \sim j$.

At each multiphase bus, the distribution network model can have: (i) grounded wye-connected loads or resources; (ii) ungrounded delta-connected loads or resources; (iii) a combination of wye- and delta-connected loads or resources at the primary side of distribution transformers; or, (iv) a combination of line-to-line and line-to-grounded-neutral loads or resources at the secondary side of distribution transformers [25]. The model (iii) above is utilized when different distribution transformers with either delta or wye connections are bundled together at the same bus for network reduction purposes (e.g., when two transformers are connected through a short line). The model (iv) above is common in, e.g., North America, for commercial buildings and center-tapped transformers at the residential level; see, e.g., the IEEE 342-node low-voltage test system [26]. Without loss of generality, we let every bus $i \in \mathcal{N}$ have three wye-connected net loads (one on each phase, with grounded neutral) and three delta-connected net loads (one across each pair of phases, ungrounded). Let $s_{\mathrm{Y},i} := [s_{\mathrm{Y},i}^a, s_{\mathrm{Y},i}^b, s_{\mathrm{Y},i}^c]^T$ denote the complex power consumptions of wye-connected net loads at bus i. Let $s_{\Delta,i} := [s_{\Delta,i}^{ab}, s_{\Delta,i}^{bc}, s_{\Delta,i}^{ca}]^T$ and $I_{\Delta,i} := [I_{\Delta,i}^{ab}, I_{\Delta,i}^{bc}, I_{\Delta,i}^{ca}]^T$ denote the power consumptions and currents of delta-connected net loads at bus i, respectively. If a particular phase of a particular type of connection does not exist at bus i, then the corresponding element of $s_{\mathrm{Y},i}$ or $s_{\Delta,i}$ ($I_{\Delta,i}$) is set to zero.

III. Semidefinite Relaxation of OPF under Extended Branch Flow Model

A. Extended Branch Flow Model

We extend the branch flow model in [10] to incorporate delta connections.[1] The resultant EBFM is given as the following.

1) Ohm's law:

$$V_i - V_j = z_{ij} I_{ij}, \quad \forall i \to j. \tag{1}$$

2) Definition of auxiliary variables:

$$\ell_{ij} = I_{ij} I_{ij}^H, \; S_{ij} = V_i I_{ij}^H, \quad \forall i \to j$$
$$X_i = V_i I_{\Delta,i}^H, \quad \forall i \in \mathcal{N}. \tag{2}$$

3) Delta-connected loads:

$$s_{\Delta,i} = \mathrm{diag}(\Gamma X_i), \quad \forall i \in \mathcal{N} \tag{3}$$

where the matrix Γ is defined as:

$$\Gamma := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

4) Power balance:

$$\sum_{k:k\to i} \mathrm{diag}(S_{ki} - z_{ki}\ell_{ki}) - \sum_{j:i\to j} \mathrm{diag}(S_{ij})$$
$$= \mathrm{diag}\left(V_i V_i^H y_i^H\right) + s_{\mathrm{Y},i} + \mathrm{diag}(X_i \Gamma), \; \forall i \in \mathcal{N}. \tag{4}$$

[1]The bus injection model can be extended in a similar way, but it is not introduced here because it was shown to be numerically less stable than the branch flow model [10].

To interpret ℓ and S, note that $\text{diag}(\ell_{ij})$ denotes the magnitude squares of three phases of current I_{ij}, and $\text{diag}(S_{ij})$ denotes the sending-end three-phase power flow on line $i \to j$. Given Γ, the three-phase delta-connected load at bus i reads:

$$\begin{aligned} s_{\Delta,i} &= \begin{bmatrix} (V_i^a - V_i^b)(I_{\Delta,i}^{ab})^* \\ (V_i^b - V_i^c)(I_{\Delta,i}^{bc})^* \\ (V_i^c - V_i^a)(I_{\Delta,i}^{ca})^* \end{bmatrix} \\ &= \text{diag}\left(\Gamma V_i I_{\Delta,i}^H\right) = \text{diag}(\Gamma X_i), \end{aligned} \tag{5}$$

and the three-phase power flow from bus i to the delta-connected load is given by:

$$\begin{bmatrix} V_i^a(I_{\Delta,i}^{ab} - I_{\Delta,i}^{ca})^* \\ V_i^b(I_{\Delta,i}^{bc} - I_{\Delta,i}^{ab})^* \\ V_i^c(I_{\Delta,i}^{ca} - I_{\Delta,i}^{bc})^* \end{bmatrix} = \text{diag}\left(V_i I_{\Delta,i}^H \Gamma\right) = \text{diag}(X_i \Gamma). \tag{6}$$

To interpret (4), note that the receiving-end three-phase power flow on line $k \to i$ is:

$$\begin{aligned} \text{diag}(V_i I_{ki}^H) &= \text{diag}\left(V_k I_{ki}^H - (V_k - V_i) I_{ki}^H\right) \\ &= \text{diag}(S_{ki} - z_{ki}\ell_{ki}), \end{aligned}$$

and $\text{diag}\left(V_i V_i^H y_i^H\right)$ represents the three-phase power flow to the shunt element at bus i.

B. Optimal Power Flow

Let $f(s_Y, s_\Delta) : \mathbb{C}^{6(n+1)} \mapsto \mathbb{R}$ capture given performance and operational objectives associated with net loads across the entire network. Under EBFM (1)–(4), we formulate the optimal power flow (OPF) problem as:

$$\begin{aligned} \textit{OPF-EBFM:} \ \min \ & f(s_Y, s_\Delta) && \text{(7a)} \\ \text{over} \ & s_{Y,i}, s_{\Delta,i}, V_i, I_{\Delta,i} \in \mathbb{C}^3, \ X_i \in \mathbb{C}^{3\times3}, \ \forall i \in \mathcal{N} \\ & I_{ij} \in \mathbb{C}^3, \ \ell_{ij}, S_{ij} \in \mathbb{C}^{3\times3}, \quad \forall i \to j \\ \text{s.t.} \ & \text{(1)–(4)} \\ & (s_{Y,i}, s_{\Delta,i}) \in \mathcal{S}_i, \quad \forall i \in \mathcal{N} && \text{(7b)} \\ & V_0 = V_0^{\text{ref}} && \text{(7c)} \\ & \underline{V}_i^\phi \le |V_i^\phi| \le \overline{V}_i^\phi, \quad \forall i \in \mathcal{N}^+, \ \forall \phi \in \Phi. && \text{(7d)} \end{aligned}$$

The objective (7a) minimizes the operating cost. The power flow equations (1)–(4) impose physical constraints to OPF. The optimization variables (s_Y, s_Δ) integrate both controllable and uncontrollable components of wye and delta-connected net loads. By properly configuring the sets $\mathcal{S}_i$ in (7b), we can specify (i) the operational constraints on the controllable net loads; (ii) the values of the uncontrollable net loads; and (iii) the case where is no load or generation units at a certain bus/phase. The substation voltage is fixed and given as V_0^{ref} in (7c). Constraints on voltage magnitudes at all the other buses are enforced by (7d).

In (7) we can deal with the case where a certain phase, say phase a, of a certain bus, say bus j, is missing. Suppose bus j is connected to bus i through line (i, j). Then phase a of line (i, j) must also be missing. In that case we enforce $s_{Y,j}^a = s_{\Delta,j}^{ab} = s_{\Delta,j}^{ca} = 0$ by properly configuring the set $\mathcal{S}_j$ in (7b). In the impedance matrix z_{ij} and the admittance matrix y_j we set all the components that are incident to phase a to be zero. As a result of the Ohm's law (1), we have $V_j^a \equiv V_i^a$ regardless of our choice of the optimization variables. Therefore in (7d), we can either set the bound for $|V_j^a|$ the same as that for $|V_i^a|$, or drop the constraint on $|V_j^a|$ by setting a very high (low) upper (lower) bound.

C. Semidefinite Relaxation of OPF

Throughout this paper we assume that f in (7a) is a convex function and that $\mathcal{S}_i$ in (7b) are convex sets for all $i \in \mathcal{N}$. Sets $\mathcal{S}_i$ are typically convex and compact for inverter-interfaced renewable sources of energy [13] and for a number of controllable loads (e.g., variable-speed drives). Then the OPF problem (7) is nonconvex only because of the quadratic equality constraints (2), (4) and the voltage-related constraint $\underline{V}_i^\phi \le |V_i^\phi|$ in (7d). In the following, we introduce our approach to obtain the convex surrogate of (7) via semidefinite relaxation.

We first reformulate (7) as the following equivalent problem, with some newly defined parameters explained after the problem formulation:

$$\begin{aligned} \textit{OPF-EBFM}': \ \min \ & f(s_Y, s_\Delta) && \text{(8a)} \\ \text{over} \ & s_{Y,i}, s_{\Delta,i} \in \mathbb{C}^3, \ X_i \in \mathbb{C}^{3\times3}, \quad \forall i \in \mathcal{N} \\ & v_i, \rho_i \in \mathbb{H}^{3\times3}, \quad \forall i \in \mathcal{N} \\ & S_{ij} \in \mathbb{C}^{3\times3}, \ \ell_{ij} \in \mathbb{H}^{3\times3}, \quad \forall i \to j \\ \text{s.t.} \ & v_j = v_i - (S_{ij} z_{ij}^H + z_{ij} S_{ij}^H) + z_{ij}\ell_{ij} z_{ij}^H, \ \forall i \to j && \text{(8b)} \\ & s_{\Delta,i} = \text{diag}(\Gamma X_i), \quad \forall i \in \mathcal{N} && \text{(8c)} \\ & \sum_{k:k\to i} \text{diag}(S_{ki} - z_{ki}\ell_{ki}) - \sum_{j:i\to j} \text{diag}(S_{ij}) \\ & = \text{diag}\left(v_i y_i^H\right) + s_{Y,i} + \text{diag}(X_i \Gamma), \quad \forall i \in \mathcal{N} && \text{(8d)} \\ & (s_{Y,i}, s_{\Delta,i}) \in \mathcal{S}_i, \quad \forall i \in \mathcal{N} && \text{(8e)} \\ & v_0 = V_0^{\text{ref}} \left(V_0^{\text{ref}}\right)^H && \text{(8f)} \\ & \underline{v}_i \le \text{diag}(v_i) \le \overline{v}_i, \quad \forall i \in \mathcal{N}^+ && \text{(8g)} \\ & v_i \succeq 0, \quad \forall i \in \mathcal{N}_{\text{leaf}} && \text{(8h)} \\ & \begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} \succeq 0, \quad \forall i \to j && \text{(8i)} \\ & \begin{bmatrix} v_i & X_i \\ X_i^H & \rho_i \end{bmatrix} \succeq 0, \quad \forall i \in \mathcal{N} && \text{(8j)} \\ & \text{rank}\,(v_i) = 1, \quad \forall i \in \mathcal{N}_{\text{leaf}} && \text{(8k)} \\ & \text{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix}\right) = 1, \quad \forall i \to j && \text{(8l)} \\ & \text{rank}\left(\begin{bmatrix} v_i & X_i \\ X_i^H & \rho_i \end{bmatrix}\right) = 1, \quad \forall i \in \mathcal{N}. && \text{(8m)} \end{aligned}$$

The two problems (7) and (8) are connected via the following variable transformation:

$$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} = \begin{bmatrix} V_i \\ I_{ij} \end{bmatrix} \begin{bmatrix} V_i \\ I_{ij} \end{bmatrix}^H, \quad \forall i \to j \tag{9a}$$

$$\begin{bmatrix} v_i & X_i \\ X_i^H & \rho_i \end{bmatrix} = \begin{bmatrix} V_i \\ I_{\Delta,i} \end{bmatrix} \begin{bmatrix} V_i \\ I_{\Delta,i} \end{bmatrix}^H, \quad \forall i \in \mathcal{N}. \tag{9b}$$

The transformation (9) is consistent with the definition of ℓ, S, X in (2), and it explains why the positive semidefinite constraints (8h)–(8j) and rank-1 constraints (8k)–(8m) must hold. Note that (8h) and (8k) are redundant given (8j) and (8m), but we still put them there to separately describe the structures of voltages and delta connections. Constraints (8b) are obtained by multiplying both sides of (1) by their conjugate transposes. Constraints (8c)–(8e) follow (3), (4), and (7b). Constraints (8f)–(8g) follow (7c)–(7d), with inequalities in (8g) treated element-wise and $\underline{v}_i$ and $\overline{v}_i$ defined as:

$$\underline{v}_i := \left[(\underline{V}_i^a)^2, (\underline{V}_i^b)^2, (\underline{V}_i^c)^2\right]^T, \quad \forall i \in \mathcal{N}$$
$$\overline{v}_i := \left[(\overline{V}_i^a)^2, (\overline{V}_i^b)^2, (\overline{V}_i^c)^2\right]^T, \quad \forall i \in \mathcal{N}.$$

The only nonconvexity of (8) lies in the rank-1 constraints (8k)–(8m). Therefore, by removing them we obtain the following SDP, which is a convex relaxation of the original OPF problem (7):

$$\begin{aligned} \textit{EBFM-SDP:}\ \min\quad & f(s_{\mathrm{Y}}, s_{\Delta}) \\ \text{over}\quad & s_{\mathrm{Y}}, s_{\Delta}, v, S, \ell, X, \rho \\ \text{s.t.}\quad & \text{(8b)–(8j)}. \end{aligned}$$

Given an optimal solution $(s_{\mathrm{Y}}, s_{\Delta}, v, S, \ell, X, \rho)$ of EBFM-SDP that also satisfies (8k)–(8m), one can recover (V, I, I_{Δ}) and therefore obtain an optimal solution of the original OPF (7). Indeed, (V, I) can be recovered from (v, S, ℓ) using [10, Algorithm 2], and then I_{Δ} can be recovered by:

$$I_{\Delta,i} = \frac{1}{\mathrm{tr}(v_i)} X_i^H V_i, \quad \forall i \in \mathcal{N}.$$

IV. Semidefinite Relaxation of OPF under Balanced Voltage Approximation

A. Balanced Voltage Approximation

The BVA model is grounded on the following assumption:

$$\frac{V_i^a}{V_i^b} \approx \frac{V_i^b}{V_i^c} \approx \frac{V_i^c}{V_i^a} \approx e^{\mathrm{j}2\pi/3}, \quad \forall i \in \mathcal{N},$$

which asserts that three-phase voltages are nearly balanced. Under this assumption, the delta-connected net load between phases ab of bus i can be approximately represented as[2]:

$$s_{\Delta,i}^{ab} = (V_i^a - V_i^b)(I_{\Delta,i}^{ab})^* = \left(1 - e^{-\mathrm{j}2\pi/3}\right) V_i^a (I_{\Delta,i}^{ab})^*.$$

This and similar derivations for other phases lead to the linear relationship between $s_{\Delta,i}$ in (5) and $\mathrm{diag}(X_i\Gamma)$ in (6):

$$\mathrm{diag}(X_i\Gamma) = \Xi s_{\Delta,i} \tag{11}$$

where

$$\Xi = \frac{\sqrt{3}}{3}\begin{bmatrix} e^{-\mathrm{j}\pi/6} & 0 & e^{\mathrm{j}\pi/6} \\ e^{\mathrm{j}\pi/6} & e^{-\mathrm{j}\pi/6} & 0 \\ 0 & e^{\mathrm{j}\pi/6} & e^{-\mathrm{j}\pi/6} \end{bmatrix}.$$

[2]For convenience we use "=" instead of "≈" even if the equality holds approximately.

By (11), we modify EBFM (1)–(4) to obtain the following approximate power flow model under BVA.

1) Ohm's law:
$$V_i - V_j = z_{ij} I_{ij}, \quad \forall i \to j. \tag{12}$$

2) Definition of auxiliary variables:
$$\ell_{ij} = I_{ij} I_{ij}^H,\ S_{ij} = V_i I_{ij}^H, \quad \forall i \to j. \tag{13}$$

3) Power balance:
$$\sum_{k:k\to i} \mathrm{diag}(S_{ki} - z_{ki}\ell_{ki}) - \sum_{j:i\to j} \mathrm{diag}(S_{ij}) = \mathrm{diag}\left(V_i V_i^H y_i^H\right) + s_{\mathrm{Y},i} + \Xi s_{\Delta,i},\ \forall i \in \mathcal{N}. \tag{14}$$

In the BVA model, the delta-connected net loads s_{Δ} contribute to the power balance equation (14) via the constant coefficient matrix Ξ, instead of relying on the auxiliary variable X.

B. OPF and Semidefinite Relaxation

Under the BVA power flow model (12)–(14), we formulate the OPF problem as the following:

$$\textit{OPF-BVA:}\ \min\ f(s_{\mathrm{Y}}, s_{\Delta}) \tag{15a}$$
$$\begin{aligned} \text{over}\quad & s_{\mathrm{Y},i}, s_{\Delta,i}, V_i \in \mathbb{C}^3, \quad \forall i \in \mathcal{N} \\ & I_{ij} \in \mathbb{C}^3,\ \ell_{ij}, S_{ij} \in \mathbb{C}^{3\times 3}, \quad \forall i \to j \\ \text{s.t.}\quad & \text{(12)–(14)} \end{aligned}$$
$$(s_{\mathrm{Y},i}, s_{\Delta,i}) \in \mathcal{S}_i, \quad \forall i \in \mathcal{N} \tag{15b}$$
$$V_0 = V_0^{\mathrm{ref}} \tag{15c}$$
$$\underline{V}_i^{\phi} \le |V_i^{\phi}| \le \overline{V}_i^{\phi}, \quad \forall i \in \mathcal{N}^+,\ \forall \phi \in \Phi. \tag{15d}$$

Similar to the way in which (7) is reformulated as (8), we obtain the following problem, which is equivalent to (15):

$$\textit{OPF-BVA}':\ \min\ f(s_{\mathrm{Y}}, s_{\Delta}) \tag{16a}$$
$$\begin{aligned} \text{over}\quad & s_{\mathrm{Y},i}, s_{\Delta,i} \in \mathbb{C}^3,\ v_i \in \mathbb{H}^{3\times 3}, \quad \forall i \in \mathcal{N} \\ & S_{ij} \in \mathbb{C}^{3\times 3},\ \ell_{ij} \in \mathbb{H}^{3\times 3}, \quad \forall i \to j \end{aligned}$$
$$\text{s.t.}\quad v_j = v_i - (S_{ij} z_{ij}^H + z_{ij} S_{ij}^H) + z_{ij}\ell_{ij} z_{ij}^H,\ \forall i \to j \tag{16b}$$
$$\sum_{k:k\to i} \mathrm{diag}(S_{ki} - z_{ki}\ell_{ki}) - \sum_{j:i\to j} \mathrm{diag}(S_{ij}) = \mathrm{diag}\left(v_i y_i^H\right) + s_{\mathrm{Y},i} + \Xi s_{\Delta,i}, \quad \forall i \in \mathcal{N} \tag{16c}$$
$$(s_{\mathrm{Y},i}, s_{\Delta,i}) \in \mathcal{S}_i, \quad \forall i \in \mathcal{N} \tag{16d}$$
$$v_0 = V_0^{\mathrm{ref}}\left(V_0^{\mathrm{ref}}\right)^H \tag{16e}$$
$$\underline{v}_i \le \mathrm{diag}(v_i) \le \overline{v}_i, \quad \forall i \in \mathcal{N}^+ \tag{16f}$$
$$v_i \succeq 0, \quad \forall i \in \mathcal{N}_{\mathrm{leaf}} \tag{16g}$$
$$\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix} \succeq 0, \quad \forall i \to j \tag{16h}$$
$$\mathrm{rank}\,(v_i) = 1, \quad \forall i \in \mathcal{N}_{\mathrm{leaf}} \tag{16i}$$
$$\mathrm{rank}\left(\begin{bmatrix} v_i & S_{ij} \\ S_{ij}^H & \ell_{ij} \end{bmatrix}\right) = 1, \quad \forall i \to j. \tag{16j}$$

Compared to (8), the problem (16) drops auxiliary variables (X, ρ) and the associated constraints that were used to represent delta connections. By removing the rank-1 constraints

(16i)–(16j), we obtain the following SDP, which is a convex relaxation of (15).

$$\begin{aligned} \textit{BVA-SDP:} \quad & \min \quad f(s_{\rm Y}, s_\Delta) \\ & \text{over} \quad s_{\rm Y}, s_\Delta, v, S, \ell \\ & \text{s.t.} \quad \text{(16b)–(16h)}. \end{aligned}$$

Given an optimal solution $(s_{\rm Y}, s_\Delta, v, S, \ell)$ of BVA-SDP that also satisfies (16i)–(16j), one can recover (V, I) using [10, Algorithm 2] and hence obtain an optimal solution of (15); however, this recovered (V, I) may not be the exact voltages and currents in the network under optimal power injections $(s_{\rm Y}, s_\Delta)$ because the power flow model (12)–(14) is approximate under BVA. One can use OpenDSS or other power flow solvers to obtain the exact voltages and currents under $(s_{\rm Y}, s_\Delta)$. In Section V-C, we will show that the recovered and exact voltages are close, implying accuracy of the BVA model.

V. Numerical Results

We assess the performance of EBFM-SDP and BVA-SDP through numerical studies on the IEEE 13- and 37-node test feeders. In particular, we check both schemes to determine 1) if they can be solved by a generic optimization solver such as SeDuMi; 2) how close their solutions are to rank one; 3) how optimal their solutions are; and 4) how close the voltages recovered from (v, S, ℓ) are to the voltages calculated by OpenDSS under the same power injections.

A. Experiment Setup

The IEEE test networks are modeled by EBFM (1)–(4) and BVA (12)–(14) with the following simplifications [10], [25]: 1) load transformers are modeled as lines with equivalent impedances; 2) switches are modeled as open or short lines depending on their status; 3) substation voltages are $V_0^{\rm ref} = \overline{V}[1, e^{-{\rm j}2\pi/3}, e^{{\rm j}2\pi/3}]^T$, where $\overline{V}$ will be specified later, and substation transformers and regulators are removed; 4) distributed load (shunt admittance) along a line is split into two identical loads (shunt admittances) at the terminal buses of this line; and 5) capacitor banks are modeled as controllable (in continuous values) reactive power sources.

We solve OPF to determine optimal decisions for demand response in distribution networks. The objective is to regulate power flow at the substation such that it tracks a reference signal from the system operator while minimizing total disutility caused by deviating from the nominal load power usage. Specifically, our objective function is:

$$f(s_{\rm Y}, s_\Delta) = w_{{\rm disu},p} D_p(p_{\rm Y}, p_\Delta) + w_{{\rm disu},q} D_q(q_{\rm Y}, q_\Delta) + w_{{\rm track},p} \frac{\left(\mathbf{1}^T p_{{\rm Y},0} - p_{\rm sub}^{\rm ref}\right)^2}{\overline{p}_{\rm sub}} + w_{{\rm track},q} \frac{\left(\mathbf{1}^T q_{{\rm Y},0} - q_{\rm sub}^{\rm ref}\right)^2}{\overline{q}_{\rm sub}} \tag{18}$$

which is explained in the following two parts.

i) Disutility for load control. With $s = p + {\rm j}q$, the functions:

$$\begin{aligned} D_p(p_{\rm Y}, p_\Delta) = & \sum_{i \in \mathcal{L}_{\rm Y}} \sum_{\phi \in \Phi_{{\rm Y},i}} \frac{1}{2\overline{p}_{{\rm Y},i}^\phi} \left(p_{{\rm Y},i}^\phi - \overline{p}_{{\rm Y},i}^\phi\right)^2 \\ & + \sum_{i \in \mathcal{L}_\Delta} \sum_{\phi' \in \Phi'_{\Delta,i}} \frac{1}{2\overline{p}_{\Delta,i}^{\phi'}} \left(p_{\Delta,i}^{\phi'} - \overline{p}_{\Delta,i}^{\phi'}\right)^2 \\ D_q(q_{\rm Y}, q_\Delta) = & \sum_{i \in \mathcal{L}_{\rm Y}} \sum_{\phi \in \Phi_{{\rm Y},i}} \frac{1}{2\overline{q}_{{\rm Y},i}^\phi} \left(q_{{\rm Y},i}^\phi - \overline{q}_{{\rm Y},i}^\phi\right)^2 \\ & + \sum_{i \in \mathcal{L}_\Delta} \sum_{\phi' \in \Phi'_{\Delta,i}} \frac{1}{2\overline{q}_{\Delta,i}^{\phi'}} \left(q_{\Delta,i}^{\phi'} - \overline{q}_{\Delta,i}^{\phi'}\right)^2 \end{aligned}$$

are disutilities associated with real and reactive power loads, respectively. In particular, $\mathcal{L}_{\rm Y}$ and $\mathcal{L}_\Delta$ denote the sets of buses with wye and delta-connected loads, respectively, and $\Phi_{{\rm Y},i} \subseteq \{a, b, c\}$ and $\Phi'_{\Delta,i} \subseteq \{ab, bc, ca\}$ denote the sets of phases at which wye and delta-connected loads exist at bus i, respectively. The $\overline{p}$ and $\overline{q}$ are nominal real and reactive power loads, which take the same values as the load data given in the IEEE test cases. The feasible regions of p and q, i.e., the sets $\mathcal{S}_i$ in (7b), (8e) are defined as $0.5\overline{p} \le p \le \overline{p}$ and $0.5\overline{q} \le q \le \overline{q}$. A capacitor bank with nominal reactive power $\overline{q}_{\rm cap}$ is treated as a controllable reactive power injection $q_{\rm cap} \in [0, \overline{q}_{\rm cap}]$, where $q_{\rm cap}$ does not incur any disutility.

ii) Power tracking at the substation. The third and fourth terms on the right-hand-side of (18) are tracking errors for real and reactive power flows at the substation, respectively. In particular, $\mathbf{1}^T p_{{\rm Y},0}$ and $\mathbf{1}^T q_{{\rm Y},0}$ add up three-phase real and reactive powers flowing upstream from the substation to the main grid, and their reference values, $p_{\rm sub}^{\rm ref}$ and $q_{\rm sub}^{\rm ref}$, are set around 80% of the additive inverses of total real and reactive power loads. We scale the tracking errors by the total real and reactive power loads $\overline{p}_{\rm sub}$ and $\overline{q}_{\rm sub}$ so that they have a similar order of magnitude with the disutility.

In our experiments, we choose weighting factors $w_{{\rm disu},p} = w_{{\rm disu},q} = 0.1$ and $w_{{\rm track},p} = w_{{\rm track},q} = 0.4$.

B. Solutions of EBFM-SDP and BVA-SDP

As described above, we formulate EBFM-SDP and BVA-SDP in the IEEE 13- and 37-node networks with different choices of $\overline{V}$ and $\underline{V}$, which are the uniform upper and lower limits of voltage magnitudes at all the buses and phases. To solve them, we call the optimization solver SeDuMi from CVX, a MATLAB-based convex modeling framework [33]. The ranks and objective values of the SDP solutions are summarized in Tables I and II, respectively. Table II also collects the elapsed CPU times when SeDuMi solves each SDP on a Dell laptop with Intel Core i5-4300 CPU (1.90 GHz, 2.50 GHz), 16 GB RAM, 64-bit Windows 7 OS, and MATLAB R2015b.

In Tables I and II, the column "voltage" specifies voltage limits $\overline{V}$ and $\underline{V}$. For example, 2% means $[\underline{V}, \overline{V}] = [0.98, 1.02]$ p.u. The column "method" specifies whether EBFM-SDP or BVA-SDP is successfully solved.

In Table I, smaller "(v, S, ℓ)-ratio" and "(v, X, ρ)-ratio" mean that the SDP solution is closer to rank one. Specifically,

TABLE I: Ranks of SDP Solutions

network	voltage	method	(v, S, ℓ)-ratio	(v, X, ρ)-ratio
IEEE 13	2%	EBFM	1.028×10^{-7}	0.9893
		BVA	2.443×10^{-7}	-
	5%	EBFM	1.194×10^{-7}	0.9577
		BVA	1.733×10^{-7}	-
IEEE 37	2%	EBFM	1.111×10^{-6}	0.9352
		BVA	8.605×10^{-3}	-
	5%	EBFM	1.155×10^{-6}	0.9094
		BVA	9.494×10^{-8}	-

TABLE II: CPU Times and Objective Values of SDPs

network	voltage	method	time (s)	Obj_{opt}	$\text{Obj}_{\text{no control}}$
IEEE 13	2%	EBFM	2.246	10.65	106.6
		BVA	1.763	10.93	
	5%	EBFM	2.200	10.55	105.0
		BVA	2.075	10.83	
IEEE 37	2%	EBFM	9.719	6.348	64.59
		BVA	5.366	7.019	
	5%	EBFM	9.438	6.271	63.42
		BVA	3.136	6.379	

to obtain the (v, S, ℓ)-ratio of an SDP solution, we compute the largest two eigenvalues λ_1, λ_2 ($\lambda_1 \geq \lambda_2 \geq 0$) of the matrices in (8l) [(16j)] for all $i \to j$ (their eigenvalues are all real and nonnegative because they are Hermitian and positive semi-definite), and take the average (over all $i \to j$) of (λ_2/λ_1). The (v, X, ρ)-ratios are computed in the same way but for the matrices in (8m). We observe from Table I that, in most cases, the (v, S, ℓ)-ratios are very small, whereas the (v, X, ρ)-ratios are very large. In other words, the SDP relaxation is numerically exact with respect to voltages and branch flows (consistent with observations in [10]), but is usually not exact with respect to the delta-connected variables. In our future work, we will investigate the reason for such non-exactness, and develop exact convex relaxation of OPF in multiphase networks with delta connections.

In Table II, the column "time" shows the elapsed CPU times to solve the SDPs, from which we see that SDP-BVA can be solved faster than SDP-EBFM. The column "obj_{opt}" shows the minimum objective values obtained by solving SDPs. A special treatment in calculating the objective values of the BVA-SDPs is that we run OpenDSS to obtain the exact substation power under the SDP-solved loads to calculate the accurate tracking error. For comparison, the column "$\text{obj}_{\text{no control}}$" shows the objective values when all the loads consume nominal power and none provide demand response. We see that both EBFM-SDP and BVA-SDP significantly decrease the OPF objective compared to the cases without control. Moreover, the difference between the objective values of EBFM-SDP and BVA-SDP gives an upper bound on the optimality gap of BVA-SDP, i.e., how far is the BVA-SDP objective to the actual minimum, because EBFM-SDP expands the feasible set of the original OPF (7). We observe that this gap is small in most cases.

Among all the cases we tested, the largest (v, S, ℓ)-ratio and optimality gap both occur with the BVA-SDP in the IEEE 37-node network, under the 0.98–1.02 p.u. voltage limit. The substantial impacts of the network configuration and voltage limit on the performance of the proposed schemes remain to be studied in our future work.

C. Accuracy of Recovered Voltages

In Table III, we compare the voltages recovered from (v, S, ℓ) using [10, Algorithm 2] and the voltages solved by OpenDSS under the same power injections at the optimal solutions of the EBFM-SDPs and BVA-SDPs. Specifically, we consider the voltages solved by OpenDSS to be exact, and show the root-mean-square-error (RMSE) and maximum absolute error (MAX), over all the buses and phases, of the recovered voltages from the exact ones.

TABLE III: Difference between Recovered and Exact Voltages

network	voltage	method	RMSE (pu)	MAX (pu)
IEEE 13	2%	EBFM	6.953×10^{-3}	1.467×10^{-2}
		BVA	1.488×10^{-4}	2.915×10^{-4}
	5%	EBFM	6.753×10^{-3}	1.424×10^{-2}
		BVA	1.357×10^{-4}	2.673×10^{-4}
IEEE 37	2%	EBFM	6.528×10^{-3}	1.113×10^{-2}
		BVA	2.547×10^{-4}	9.833×10^{-4}
	5%	EBFM	6.343×10^{-3}	1.081×10^{-2}
		BVA	2.855×10^{-4}	5.496×10^{-4}

We observe from Table III that the voltages recovered from the solutions of the BVA-SDPs stay close to the exact voltages solved by OpenDSS. This implies the accuracy of the approximate relationship (11) that leads to the BVA model (12)–(14). The voltages recovered from the solutions of the EBFM-SDPs, however, deviate further from the exact voltages. This is not surprising given the fact that the solutions of the EBFM-SDPs, in terms of the delta-connected variables (v, X, ρ), are far from rank one.

VI. Conclusions and Future Work

Two power flow models, EBFM and BVA, were introduced for multiphase radial networks with wye and delta connections. Under both models, we formulated OPF problems and developed their SDP relaxations. Numerical studies on IEEE 13- and 37-node networks showed that both SDP relaxations can be efficiently solved by SeDuMi, whereas BVA-SDP is solved faster than EBFM-SDP. Moreover, both SDP solutions are close to rank one, and hence numerically exact with respect to voltages and branch flows. The SDP solution under EBFM is far from rank one in terms of the delta-connected variables, and therefore does not qualify as a good solution itself; however, it provides a lower bound of the OPF objective and reveals a small optimality gap of the SDP solution under BVA. Finally, by comparing the voltages recovered from the SDP solution and solved by OpenDSS, we saw that BVA is accurate in the sense that it reproduces actual system voltages.

To strengthen our findings, we will perform numerical tests on other IEEE test feeders as well as realistic systems.

Moreover, we plan to develop and test scalable distributed algorithms to solve the proposed SDP relaxations while borrowing ideas from existing algorithms based on primal-dual iterations [34]; alternate direction method of multipliers [6], [7]; interior point method [35]; auxiliary problem principle [36]; predictor-corrector proximal multiplier [1], etc. We are also interested in studying formulations and conditions for the exact convex relaxation of OPF in multiphase radial networks, which is generally believed to be a hard problem.

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